

Comment on “Critical point scaling of Ising spin glasses in a magnetic field” by J. Yeo and M.A. Moore

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In a section of a recent publication, [J. Yeo and M.A. Moore, Phys. Rev. B **91**, 104432 (2015)], the authors discuss some of the arguments in the paper by Parisi and Temesvári [Nuclear Physics B **858**, 293 (2012)]. In this comment, it is shown how these arguments are misinterpreted, and the existence of the Almeida-Thouless transition *in* the upper critical dimension 6 reasserted.

In a recent paper¹ by Yeo and Moore about the long debated existence of the Almeida-Thouless instability² in the short ranged Ising spin glass below the upper critical dimension six, the authors criticize in Sec. III. some of our statements and arguments in Ref.³. In that paper we have demonstrated: firstly the incorrect reasoning of Ref.⁴ about the disappearance of the Almeida-Thouless (AT) transition line when approaching the upper critical dimension from above; secondly we have computed the AT line staying exactly in six dimensions (and not by a limiting process); and thirdly the ϵ -expansion was used to compute the AT line below six dimensions, and the relatively smooth behavior of it while crossing $d = 6$ (with fixed bare parameters) was exhibited. In what follows, we want to comment the discussion in Sec. III. of¹.

I. AT AND ABOVE SIX DIMENSIONS

The first order renormalization group (RG) equations for the six-dimensional model are worked out and solved in Sec. 3 of Ref.³, the AT line follows from that calculation [see Eq. (37) in³]¹:

$$h_{\text{AT}}^2 = \frac{4}{(1 - w^2 \ln |r| + \frac{10}{3} w^2 \ln w)^4} w |r|^2 \approx \frac{4}{(1 - w^2 \ln |r|)^4} w |r|^2, \quad d = 6 \quad (1)$$

where $w^2 \ll 1$ was used. [Note that a minus sign in the denominator of Eq. (13) has been left out in¹.] As it turns out from the discussion in Sec. 3 of Ref.³, this approximation is valid if the scaling variable with zero scaling dimension (which is invariant under RG in $d = 6$) is small, i.e.

$$\frac{w^2}{1 + \frac{5}{3} w^2 \ln w^2 - w^2 \ln |r|} \ll 1, \quad (2)$$

and this condition is always satisfied whenever $|r| \ll 1$ and $w^2 \ll 1$; see also the middle part of Eq. (59) of that reference. Yeo and Moore¹ forget all about this derivation of the six-dimensional AT line; they deduce it from Eq. (11) of¹ by the limit $\epsilon \rightarrow 0$, and finally they argue that “Eq. (11) is not valid for this limit”. We can absolutely agree with this last statement: the system at the upper critical dimension needs special care, physical quantities, like the critical magnetic field where replica symmetry breaking sets in, cannot be obtained by a limiting process of $\epsilon \rightarrow 0$. The point is that ϵ in Eq. (11) may be small, but fixed, while $|r| \ll 1$, and the $|r|^{\epsilon/2}$ term in the denominator must be ignored. Taking account of this, the AT line above dimension six, Eq. (11) of¹, must be written (consistently with the approximations used to derive it) as:

$$h_{\text{AT}}^2 \sim \frac{w |r|^{\frac{d}{2}-1}}{(\frac{2w^2}{\epsilon} + 1)^{\frac{5d}{6}-1}}, \quad d > 6. \quad (3)$$

This is just Eq. (28) of Ref.³. This equation for the AT line above six dimensions must be supplemented by the range of its applicability, otherwise false conclusions like Eq. (12) in¹ [which is obviously incompatible with (1)] could be deduced. For this reason, we briefly repeat the two steps needed for the derivation of (3):

¹ We use here the notations of Ref.¹. In fact $|r|$ was called τ in³, whereas r had the role of the nonlinear scaling field associated with τ . We also adapt here to the somewhat unconventional use of the symbol ϵ as $\epsilon = d - 6$.

- The RG equations for the three bare parameters, namely

$$\begin{aligned} |\dot{r}| &= \left(2 - \frac{10}{3}w^2\right) |r|, \\ \dot{w}^2 &= -\epsilon w^2 - 2w^4, \\ \dot{h}^2 &= \left(4 + \frac{\epsilon}{2} + \frac{1}{3}w^2\right) h^2 \end{aligned} \quad (4)$$

are valid for $|r| \ll 1$ and $w^2 \ll 1$. One can introduce the nonlinear scaling fields⁵ satisfying exactly, by definition, the linearized (around the fixed point) and diagonalized RG equations. For the system in (4), and for its Gaussian fixed point, one readily finds

$$g_{|r|} = 2 g_{|r|}, \quad g_{w^2} = -\epsilon g_{w^2}, \quad \text{and} \quad g_{h^2} = \left(4 + \frac{\epsilon}{2}\right) g_{h^2}.$$

The relations between bare parameters and nonlinear scaling fields were published in³, for completeness we repeat them here:

$$|r| = g_{|r|} \left(1 - \frac{2}{\epsilon} g_{w^2}\right)^{-\frac{5}{3}}, \quad w^2 = g_{w^2} \left(1 - \frac{2}{\epsilon} g_{w^2}\right)^{-1}, \quad \text{and} \quad h^2 = g_{h^2} \left(1 - \frac{2}{\epsilon} g_{w^2}\right)^{\frac{1}{6}}. \quad (5)$$

- The zeros of the scaling function of the replicon mass, $\hat{\Gamma}_R$, are the locations of the AT instability. $\hat{\Gamma}_R$ depends on the bare parameters $|r|$, w^2 , and h^2 through the RG invariants $x \equiv g_{w^2} g_{|r|}^{\frac{\epsilon}{2}}$ and $y \equiv g_{h^2} g_{|r|}^{-2-\frac{\epsilon}{4}}$. The AT instability line can then be written as $y = f(x)$ or

$$g_{h_{\text{AT}}^2} = g_{|r|}^{2+\frac{\epsilon}{4}} f\left(g_{w^2} g_{|r|}^{\frac{\epsilon}{2}}\right) = \frac{g_{|r|}^2}{\sqrt{g_{w^2}}} g\left(g_{w^2} g_{|r|}^{\frac{\epsilon}{2}}\right), \quad \text{with} \quad g(x) \equiv \sqrt{x} f(x). \quad (6)$$

The following remarks are now in order:

- (i) This form of the AT line is generic for the system where the zero-external-magnetic-field symmetry is broken only by the linear replica symmetric invariant in the Lagrangian whose bare coupling constant is h^2 . (This model is used in Refs.^{1,4} too.) Eqs. (5) cannot be used, in this generic case, to replace nonlinear scaling fields by bare couplings, as they were derived from the one-loop RG equations in (4).
- (ii) Eq. (14) of¹ formally agrees with (6), but the bare couplings are there instead of the g 's. In this form it is not correct.
- (iii) The function $g(x)$ of (6) can be calculated perturbatively, the 1-loop result was published in³: $g(x) = (-C')x$ where $-C'(\epsilon) > 0$ is analytic and positive around $\epsilon = 0$. Putting this into (6), one gets

$$g_{h_{\text{AT}}^2} \sim g_{|r|}^{2+\frac{\epsilon}{2}} \sqrt{g_{w^2}},$$

and inserting the inverse relations of those in Eq. (5) one immediately arrives at (3).

As it must be clear from the two-step process above, a mixture of renormalization *and* perturbation theory leads to Eq. (3). The leading, linear contribution to $g(x)$ is free from a singularity at $d = 6$, as it comes from an ultraviolet convergent one-loop graph³. Triangular insertions in the next, two-loop graphs, however, certainly produce singular terms like $g(x) \sim \frac{1}{\epsilon} x^2$, their neglect is acceptable only if $\frac{1}{\epsilon} x = \frac{1}{\epsilon} g_{w^2} g_{|r|}^{\frac{\epsilon}{2}} \ll 1$. Expressing this condition by the bare couplings, one can write the range of applicability of Eq. (3) as

$$|r| \ll 1, \quad w^2 \ll 1, \quad \text{and most importantly} \quad \frac{1}{\epsilon} w^2 |r|^{\frac{\epsilon}{2}} \left(1 + \frac{2}{\epsilon} w^2\right)^{-1-\frac{5}{6}\epsilon} \ll 1. \quad (7)$$

The left-hand-side of the third condition becomes of order unity (1/2), and thus breaks down, when $\epsilon \rightarrow 0$ while $|r|$ and $w^2 \ll 1$, but otherwise fixed. This is just the limit leading to Eq. (12) of¹ (and to the conclusion of the disappearance of the AT line for $\epsilon \rightarrow 0$), and is the source of the basic fault in the original arguments in⁴. [See also Fig. 2(b) and the discussion around it in³.] ϵ in (3) may be small, but must be kept fixed. Simple first order perturbational result is obtained for $w^2 \ll \epsilon$. The joint application of the perturbational method and RG (and not

RG alone as Yeo and Moore¹ claim) provide (3) which is valid for $0 < \epsilon \ll w^2 \ll 1$ too. In this latter case the range of applicability of Eq. (3), according to (7), shrinks to zero as $-\ln|r| \gg \epsilon^{-1}$, together with the amplitude in (3). This phenomenon signals the appearance of the logarithmic correction in $d = 6$: $h_{\text{AT}}^2 \sim (\ln|r|)^{-4} |r|^2$, and it is not an indication of the disappearance of the AT line.

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¹ J. Yeo and M. Moore, Phys. Rev. B **91**, 104432 (2015), arXiv:1412.2448.

² J. R. L. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).

³ G. Parisi and T. Temesvári, Nucl. Phys. B **858**, 293 (2012), arXiv:1111.3313.

⁴ M. Moore and A. Bray, Phys. Rev. B **83**, 224408 (2011), arXiv:1102.1675.

⁵ F. J. Wegner, Phys. Rev. B **5**, 4529 (1972).